SHARP BOUNDS FOR NEUMAN-SÁNDOR'S MEAN IN TERMS OF THE ROOT-MEAN-SQUARE

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Abstract. In the paper, the authors find sharp bounds for Neuman-Sándor's mean in terms of the root-mean-square.

1. Introduction

Throughout this paper, we assume that the real numbers a and b are positive and that $a \ne b$.

The second Seiffert's mean $T(a,b)$ and Neuman-Sándor's mean $M(a,b)$ are respectively defined in [5, 7] by

$$T(a,b)=\frac{a-b}{2\arctan\bigl(\frac{a-b}{a+b}\bigr)} \quad \text{and} \quad M(a,b)=\frac{a-b}{2\operatorname{arcsinh}\bigl(\frac{a-b}{a+b}\bigr)}, \tag{1.1}$$

while the arithmetic mean and the root-mean-square are respectively defined by

$$A(a,b)=\frac{a+b}{2} \quad \text{and} \quad S(a,b)=\sqrt{\frac{a^2+b^2}{2}}\,. \tag{1.2}$$

A chain of inequalities between these four means

$$A<M<T<S$$

were established in [5, 6].

In [2], the authors demonstrated that the double inequality

$$\alpha S(a,b)+(1-\alpha)A(a,b)<T(a,b)<\beta S(a,b)+(1-\beta)A(a,b)$$

holds if and only if $\alpha \le \frac{4-\pi}{(\sqrt{2}-1)\pi}$ and $\beta \ge \frac{2}{3}$.

In [3, 4], the authors independently found that the double inequality

$$\alpha S(a,b)+(1-\alpha)A(a,b)<M(a,b)<\beta S(a,b)+(1-\beta)A(a,b)$$

holds if and only if $\alpha \le \frac{1-\ln(1+\sqrt{2}\,)}{(\sqrt{2}-1)\ln(1+\sqrt{2}\,)}$ and $\beta \ge \frac{1}{3}$.

In [1], the authors discovered that the double inequality

$$S\bigl(\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a\bigr)<T(a,b)<S\bigl(\beta a+(1-\beta)b,\beta b+(1-\beta)a\bigr)$$

holds if and only if $\alpha \le \frac{1+\sqrt{16/\pi^2-1}}{2}$ and $\beta \ge \frac{3+\sqrt{6}}{6}$.

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Motivated by the above double inequalities, we naturally ask a question: What are the best constants $\alpha \ge \frac{1}{2}$ and $\beta \le 1$ such that the double inequality

$$S\big(\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a\big) < M(a,b) < S\big(\beta a+(1-\beta)b,\beta b+(1-\beta)a\big) \tag{1.3}$$

holds for all $a,b>0$ with $a\ne b$?

The aim of this paper is just to give an affirmative answer to this question.

The main result of this paper may be formulated as the following theorem.

Theorem 1.1. *The double inequality* (1.3) *holds true if and only if*

$$\alpha \le \frac{1}{2}\left\{1+\sqrt{\frac{1}{\big[\ln\big(1+\sqrt{2}\,\big)\big]^2}-1}\right\} = 0.76\dots \quad \textit{and} \quad \beta \ge \frac{3+\sqrt{3}}{6} = 0.78\dots.$$

2. Proof of Theorem 1.1

For simplicity, denote

$$\lambda = \frac{1}{2}\left\{1+\sqrt{\frac{1}{\big[\ln\big(1+\sqrt{2}\,\big)\big]^2}-1}\right\} \quad \text{and} \quad \mu = \frac{3+\sqrt{3}}{6}.$$

It is clear that, in order to prove the double inequality (1.3), it suffices to show

$$M(a,b) > S\big(\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a\big) \tag{2.1}$$

and

$$M(a,b) < S\big(\mu a+(1-\mu)b,\mu b+(1-\mu)a\big). \tag{2.2}$$

From definitions in (1.1) and (1.2), we see that both $M(a,b)$ and $S(a,b)$ are symmetric and homogeneous of degree 1. Hence, without loss of generality, we assume that $a>b>0$. If replacing $\frac{a}{b}>1$ by $t>1$ and letting $p\in\big(\frac{1}{2},1\big)$, then

$$\begin{aligned} &S(pa+(1-p)b,pb+(1-p)a)-M(a,b)\\ &\qquad = \frac{b}{2}\frac{\sqrt{[pt+(1-p)]^2+[p+(1-p)t]^2}}{\operatorname{arcsinh}\frac{t-1}{t+1}}f(t), \end{aligned} \tag{2.3}$$

where

$$f(t)=\sqrt{2}\operatorname{arcsinh}\frac{t-1}{t+1}-\frac{t-1}{\sqrt{[pt+(1-p)]^2+[p+(1-p)t]^2}}. \tag{2.4}$$

Standard computations lead to

$$f(1)=0, \tag{2.5}$$

$$\lim_{t\to\infty} f(t)=\sqrt{2}\,\ln\big(1+\sqrt{2}\,\big)-\frac{1}{\sqrt{2p^2-2p+1}}, \tag{2.6}$$

and

$$f'(t)=\frac{f_1(t)}{(1+t)\sqrt{1+t^2}\,\{[pt+(1-p)]^2+[p+(1-p)t]^2\}^{3/2}}, \tag{2.7}$$

where

$$f_1(t)=2\big\{[pt+(1-p)]^2+[p+(1-p)t]^2\big\}^{3/2}-(1+t)^2\sqrt{1+t^2} \tag{2.8}$$

and

$$\Big\{2\big([pt+(1-p)]^2+[p+(1-p)t]^2\big)^{3/2}\Big\}^2-\Big[(1+t)^2\sqrt{1+t^2}\,\Big]^2=(t-1)^2g_1(t) \tag{2.9}$$

with

$$
\begin{aligned}
g_1(t) = {} & \bigl(32p^6 - 96p^5 + 144p^4 - 128p^3 + 72p^2 - 24p + 3\bigr)t^4 \\
& - 2\bigl(64p^6 - 192p^5 + 240p^4 - 160p^3 + 48p^2 - 1\bigr)t^3 \\
& + 6\bigl(32p^6 - 96p^5 + 112p^4 - 64p^3 + 24p^2 - 8p + 1\bigr)t^2 \\
& - 2\bigl(64p^6 - 192p^5 + 240p^4 - 160p^3 + 48p^2 - 1\bigr)t \\
& + 32p^6 - 96p^5 + 144p^4 - 128p^3 + 72p^2 - 24p + 3
\end{aligned}
\tag{2.10}
$$

and

$$
g_1(1) = 16\bigl(6p^2 - 6p + 1\bigr). \tag{2.11}
$$

Let

$$
g_2(t) = \frac{g_1'(t)}{2}, \quad g_3(t) = \frac{g_2'(t)}{6}, \quad \text{and} \quad g_4(t) = g_3'(t).
$$

Then simple computations result in

$$
\begin{aligned}
g_2(t) = {} & 2\bigl(32p^6 - 96p^5 + 144p^4 - 128p^3 + 72p^2 - 24p + 3\bigr)t^3 \\
& - 3\bigl(64p^6 - 192p^5 + 240p^4 - 160p^3 + 48p^2 - 1\bigr)t^2 \\
& + 6\bigl(32p^6 - 96p^5 + 112p^4 - 64p^3 + 24p^2 - 8p + 1\bigr)t \\
& - \bigl(64p^6 - 192p^5 + 240p^4 - 160p^3 + 48p^2 - 1\bigr),
\end{aligned}
\tag{2.12}
$$

$$
g_2(1) = 16\bigl(6p^2 - 6p + 1\bigr), \tag{2.13}
$$

$$
\begin{aligned}
g_3(t) = {} & \bigl(32p^6 - 96p^5 + 144p^4 - 128p^3 + 72p^2 - 24p + 3\bigr)t^2 \\
& - \bigl(64p^6 - 192p^5 + 240p^4 - 160p^3 + 48p^2 - 1\bigr)t \\
& + 32p^6 - 96p^5 + 112p^4 - 64p^3 + 24p^2 - 8p + 1,
\end{aligned}
\tag{2.14}
$$

$$
g_3(1) = 16p^4 - 32p^3 + 48p^2 - 32p + 5, \tag{2.15}
$$

$$
\begin{aligned}
g_4(t) = {} & 2\bigl(32p^6 - 96p^5 + 144p^4 - 128p^3 + 72p^2 - 24p + 3\bigr)t \\
& - \bigl(64p^6 - 192p^5 + 240p^4 - 160p^3 + 48p^2 - 1\bigr),
\end{aligned}
\tag{2.16}
$$

$$
g_4(1) = 48p^4 - 96p^3 + 96p^2 - 48p + 7. \tag{2.17}
$$

When $p = \lambda$, the quantities (2.6), (2.11), (2.13), (2.15), and (2.17) become

$$
\lim_{t \to \infty} f(t) = 0, \tag{2.18}
$$

$$
g_1(1) = g_2(1) = -\frac{8\bigl\{4\bigl[\ln\bigl(1+\sqrt{2}\,\bigr)\bigr]^2 - 3\bigr\}}{\bigl[\ln\bigl(1+\sqrt{2}\,\bigr)\bigr]^2} < 0, \tag{2.19}
$$

$$
g_3(1) = -\frac{7\bigl[\ln\bigl(1+\sqrt{2}\,\bigr)\bigr]^4 - 4\bigl[\ln\bigl(1+\sqrt{2}\,\bigr)\bigr]^2 - 1}{\bigl[\ln\bigl(1+\sqrt{2}\,\bigr)\bigr]^4} < 0, \tag{2.20}
$$

$$
g_4(1) = -\frac{5\bigl[\ln\bigl(1+\sqrt{2}\,\bigr)\bigr]^4 - 3}{\bigl[\ln\bigl(1+\sqrt{2}\,\bigr)\bigr]^4} < 0, \tag{2.21}
$$

and

$$
32p^6 - 96p^5 + 144p^4 - 128p^3 + 72p^2 - 24p + 3 = -\frac{2\bigl[\ln\bigl(1+\sqrt{2}\,\bigr)\bigr]^6 - 1}{\bigl[2\ln\bigl(1+\sqrt{2}\,\bigr)\bigr]^6} > 0. \tag{2.22}
$$

Consequently, from (2.10), (2.12), (2.14), (2.16), and (2.18), it is very easy to obtain that

$$\lim_{t\to\infty} g_1(t)=\infty,\quad \lim_{t\to\infty} g_2(t)=\infty,\quad \lim_{t\to\infty} g_3(t)=\infty,\quad \lim_{t\to\infty} g_4(t)=\infty. \tag{2.23}$$

From (2.16) and (2.22), it is immediate to derive that the function $g_4(t)$ is strictly increasing on $[1,\infty)$, and so, by virtue of (2.21) and the final limit in (2.23), there exists a point $t_0>1$ such that $g_4(t)<0$ on $[1,t_0)$ and $g_4(t)>0$ on (t_0,∞). Hence, the function $g_3(t)$ is strictly decreasing on $[1,t_0]$ and strictly increasing on $[t_0,\infty)$. Similarly, by (2.20) and the third limit in (2.23), there exists a point $t_1>t_0>1$ such that $g_2(t)$ is strictly decreasing on $[1,t_1]$ and strictly increasing on $[t_1,\infty)$. Further, by (2.19) and the second limit in (2.23), there exists a point $t_2>t_1>1$ such that $g_1(t)$ is strictly decreasing on $[1,t_2]$ and strictly increasing on $[t_2,\infty)$. Thereafter, by (2.7) to (2.9), (2.19), and the first limit in (2.23), there exists a point $t_3>t_2>1$ such that $f(t)$ is strictly decreasing on $[1,t_3]$ and strictly increasing on $[t_3,\infty)$. As a result, the inequality (2.1) follows from equations (2.3) to (2.5), and (2.18), together with the piecewise monotonicity of $f(t)$.

When $p=\mu$, the equation (2.10) becomes

$$g_1(t)=\frac{5t^2+8t+5}{27}(t-1)^2>0 \tag{2.24}$$

for $t>1$. By equations (2.7) to (2.10) and the inequality (2.24), it can be concluded that $f(t)$ is strictly increasing and positive on $[1,\infty)$. The inequality (2.2) follows.

It is not difficult to verify that the mean $S\bigl(xa+(1-x)b,xb+(1-x)a\bigr)$ is continuous and strictly increasing on $\left[\frac12,1\right]$. From this monotonicity and inequalities (2.1) and (2.2), one can conclude that the double inequality (1.3) holds true for all $\alpha\le\lambda$ and $\beta\ge\mu$.

For any given number p satisfying $1>p>\lambda$, it is obvious that the limit (2.6) is positive. This positivity, together with (2.3) and (2.4), implies that for $1>p>\lambda$ there exists $T_0=T_0(p)>1$ such that the inequality

$$S(pa+(1-p)b,pb+(1-p)a)>M(a,b)$$

holds for $\frac{a}{b}\in(T_0,\infty)$. This tells us that the constant λ is the best possible.

For $\frac12<p<\mu$, from (2.11) one has

$$g_1(1)=16(6p^2-6p+1)<0. \tag{2.25}$$

From the inequality (2.25) and the continuity of $g_1(t)$, there exists a number $\delta=\delta(p)>0$ such that the function $g_1(t)$ is negative on $(1,1+\delta)$. This negativity, together with (2.3), (2.5), (2.7), and (2.10), implies that for any $\frac12<p<\mu$, there exists $\delta=\delta(p)>0$ such that the inequality

$$M(a,b)>S(pa+(1-p)b,pb+(1-p)a)$$

is valid for $\frac{a}{b}\in(1,1+\delta)$. Consequently, the number μ is the best possible. The proof of Theorem 1.1 is complete.

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